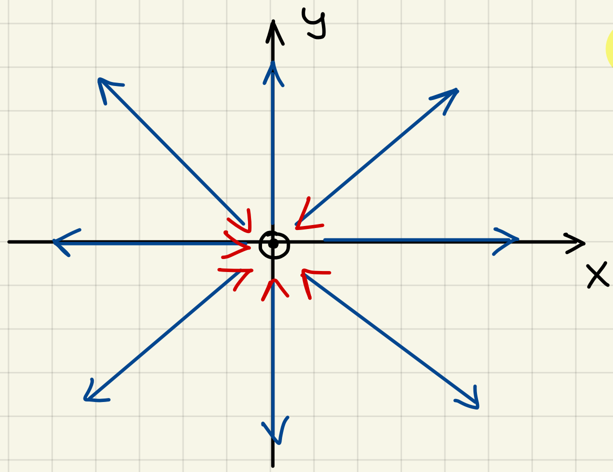
IV. Aerofoil Theory

4.2 The velocity field

$$u_r = \frac{Q}{2\pi r}, \quad u_\theta = 0,$$

where Q is a constant, is called a **line source** flow if $Q > 0$ and a **line sink** if $Q < 0$. Show that it is irrotational and incompressible, save at $r = 0$, where it is not defined. Find the velocity potential and the stream function, and show that the complex potential is

$$w = \frac{Q}{2\pi} \log z .$$



Line source: coincides with z-axis and emits fluid isotropically at the stedy rate Q — the flow is directed away from $r=0$.

Line sink: absorbs fluid at a rate Q.

$$\bar{\nabla} \times \bar{u} = 0 \quad - \text{irrot.}$$

$$\bar{\nabla} \cdot \bar{u} = u_r' + \frac{1}{r} u_r = -\frac{Q}{2\pi r^2} + \frac{Q}{2\pi r^2} = 0$$

incompr.

1) Veloc. potential $(\bar{\nabla} \times \bar{u})$

$$\bar{u} = \bar{\nabla} \varphi$$

$$u_r = \frac{\partial \varphi}{\partial r}, \quad u_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta} = 0$$

$$\rightarrow \varphi = f(r) = \frac{Q}{2\pi} \log r + \cancel{c}$$

2) Stream function $(\bar{\nabla} \cdot \bar{u})$

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}$$

$$\frac{D\psi}{Dt} = (\bar{u} \cdot \bar{\nabla}) \psi = \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y}\right) = 0$$

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r} = 0$$

$$\psi = f(\theta) = \frac{Q}{2\pi}\theta$$

3) Complex potential

$$u_x = \frac{\partial \psi}{\partial y} = \frac{\partial \varphi}{\partial x} \quad , \quad u_y = -\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial y} ,$$

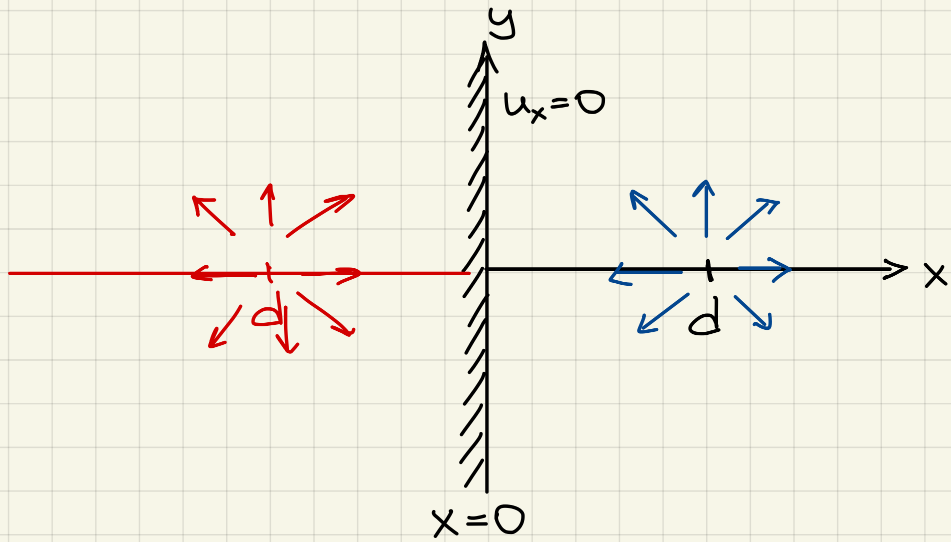
Cauchy-Riemann eq-s.

$$\underbrace{w(z) = \varphi + i\psi}_{\text{complex potential}}$$

$$w(z) = \frac{Q}{2\pi}(\log r + i\theta) = \frac{Q}{2\pi}\log z .$$

$$z = re^{i\theta} , \quad \log z = \log r + i\theta$$

Fluid occupies the region $x > 0$, and there is a plane rigid boundary at $x = 0$. Find the complex potential for the flow due to a line source at $z = d > 0$, and show that the pressure at $x = 0$ decreases to a minimum at $|y| = d$ and thereafter increases with $|y|$.



BC: $u_x = 0$

$\Rightarrow \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} = 0$

$$w(z) = w_1(z) + w_2(z)$$

$$w_1(z) = \frac{Q}{2\pi} \log(x_1 + iy) = \frac{Q}{2\pi} \log(x - d + iy) = \frac{Q}{2\pi} \log(z-d)$$

$$w_2(z) = \frac{Q}{2\pi} \log(x_2 + iy) = \frac{Q}{2\pi} \log(z+d)$$

$$w(z) = \frac{Q}{2\pi} [\log(z-d) + \log(z+d)]$$

Analyze p at $x=0$.

$x = 0$ is a streamline ($u_x = 0$, $u_y \neq 0$)

Bern. streamline th.: ideal fluid in a steady flow:

$$\frac{p}{\rho} + \frac{1}{2} \bar{u}^2 = \text{const along a streamline}$$

$$w'(z) = \frac{\partial \varphi}{\partial x} + i \frac{\partial \psi}{\partial x} = u - iv$$

$$w'(z) = \frac{Q}{2\pi} \left(\frac{1}{z-d} + \frac{1}{z+d} \right)$$

$$= \frac{Q}{2\pi} \left(\frac{1}{iy-d} + \frac{1}{iy+d} \right)$$

$$= \frac{Q}{2\pi}\left(\frac{1}{iy-d}\,\frac{iy+d}{iy+d} + \frac{1}{iy+d}\,\frac{iy-d}{iy-d}\right)$$

$$= \frac{Q}{2\pi}\,\frac{-2iy}{y^2+d^2}$$

$$\Rightarrow \underset{\text{at } x=0}{u=0}, \quad v = +\frac{Q}{\pi}\,\frac{y}{y^2+d^2}$$

$$p + \frac{\rho}{2}\,\frac{Q^2}{\pi^2}\,\frac{y^2}{(y^2+d^2)^2} = \text{const}$$

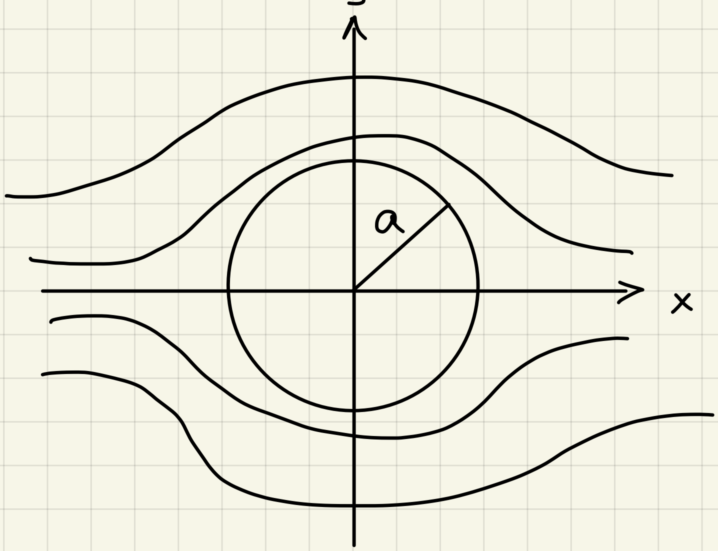
Find extrema of p:

$$\frac{dp}{dy} = -\frac{Q^2}{\pi^2}\,\frac{\rho}{2}\,\frac{d}{dy}\,\frac{y^2}{(y^2+d^2)^2} = 0$$

4.3 An irrotational 2D flow has stream function

$$\psi = A(x-c)y,$$

where A and c are constants. A circular cylinder of radius a is introduced, its centre being at the origin. Find the complex potential, and hence the stream function, of the resulting flow. Use Blasius' theorem to calculate the force exerted on the cylinder.



$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\Rightarrow \varphi = \int A(x-c)\,dx = \frac{A}{2}x^2 - Acx + f(y) + C_1$$

$$\varphi = -\int Ay\,dy = -\frac{A}{2}y^2 + g(x) + C_2$$

$$\Rightarrow \varphi(x,y) = \frac{A}{2}(x^2 - y^2) - Acx + \cancel{\tilde{C}}$$

In the abs. of a cyl:

$$w(z) = \varphi + i\psi = \frac{A}{2}(x^2 - y^2) - Acx + iA(x-c)y$$

$$= \frac{A}{2}z^2 - Acz,$$

$$z^2 = x^2 + 2ixy - y^2$$

Milne-Thomson's circle theorem: suppose we have a flow with complex potential $w = f(z)$, where all the singularities of $f(z)$ lie in $|z| > a$. Then

$$w = f(z) + \overline{f(a^2/\bar{z})}$$

is the complex potential of a flow with (I) the same singularities as $f(z)$ in $|z| > a$ and (II) $|z| = a$ as a streamline.

$$w\left(\frac{a^2}{\bar{z}}\right) = \overline{\frac{A}{2}\frac{a^4}{z^2}} - \overline{Ac\frac{a^2}{z}} = \frac{A}{2}\frac{a^4}{\underbrace{x^2 - 2ixy - y^2}_{(x-iy)^2}} - Ac\frac{a^2}{x-iy}$$

$$\frac{1}{(x-iy)^2} = \frac{1}{(x-iy)^2}\frac{(x+iy)(x+iy)}{(x+iy)(x+iy)} = \frac{z^2}{(x^2+y^2)^2}$$

$$w\left(\frac{a^2}{\bar{z}}\right) = \frac{A}{2}\frac{a^4 z^2}{(x^2+y^2)^2} - Ac\frac{a^2 z}{x^2+y^2}$$

$$\overline{w\left(\frac{a^2}{\bar{z}}\right)} = \frac{A}{2}\frac{a^4}{(x^2+y^2)^2}(x^2 - 2ixy - y^2) - Ac\frac{a^2}{x^2+y^2}(x-iy)$$

Complex pot. of a flow around the cyl.:

$$w(z) = \frac{A}{2}(x^2 + 2ixy - y^2) - Ac(x+iy) + \frac{A}{2}\frac{a^4}{(x^2+y^2)^2}(x^2 - 2ixy - y^2) - Ac\frac{a^2}{x^2+y^2}(x-iy)$$

$$\psi = \mathrm{Im}(w)$$

$$\begin{aligned}\psi &= Axy - Acy - A\frac{a^4}{(x^2+y^2)^2}xy + Ac\frac{a^2}{x^2+y^2}y \\ &= Ay\left(x - \frac{a^4}{(x^2+y^2)^2}x - c + c\frac{a^2}{x^2+y^2}\right) \\ &= Ay\left(1 - \frac{a^2}{x^2+y^2}\right)\left[x\left(1 + \frac{a^2}{x^2+y^2}\right) - c\right]\end{aligned}$$

Blasius' theorem: let there be a steady flow with complex potential *w(z)* about some fixed body which has as its boundary the closed contour *C*. If *Fx* and *Fy* are the components of the net force on the body, then

$$F_x - i F_y = \frac{1}{2} i \rho \oint_C \left(\frac{dw}{dz}\right)^2 dz .$$

$C \Rightarrow |z| = a^2$

Use Residue theorem :

$$\oint_C f(z)\,dz = 2\pi i \sum_k \operatorname{Res}(f, a_k)$$

4.4 Show that the problem of irrotational flow past a circular cylinder may be formulated in terms of the velocity potential as follows:

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0,$$

with

$$\varphi = Ur\cos\theta \text{ as } r \to \infty, \quad \frac{\partial \varphi}{\partial r} = 0 \text{ on } r = a,$$

and obtain the solution by using the method of separation of variables.

$$\bar{\nabla}\cdot\bar{u} = \bar{\nabla}^2\varphi = 0$$

At $r = a$, $u_r = 0 \rightarrow \frac{\partial \varphi}{\partial r} = 0$

At $r \to \infty$, $\bar{u} = \underbrace{(U, 0, 0)}_{\text{cartesian}}$ – free-stream

$$\bar{u} = U\bar{e}_x = U(\cos\theta\,\bar{e}_r - \sin\theta\,\bar{e}_\theta)$$

$$\varphi = \int u_r\,dr = Ur\cos\theta \quad \text{at } r \to \infty$$

$$\varphi(r,\theta) = f(r)\,g(\theta)$$

$$f''g + \frac{1}{r}f'g + \frac{1}{r^2}fg'' = 0$$

$$r^2\frac{f''}{f} + r\frac{f'}{f} = -\frac{g''}{g} = -K$$

I. $g''(\theta) - K\,g(\theta) = 0$

Non-triv. if $K < 0$, $K = -\lambda^2$

$$g(\theta) = C_1\cos\lambda\theta + C_2\sin\lambda\theta$$

At $r \to \infty$, $\varphi = Ur\cos\theta \Rightarrow C_2 = 0,\ \lambda = 1,\ C_1 = 1$

$$\Rightarrow \varphi(r,\theta) = f(r)\cos\theta$$

II. $f'' + \frac{1}{r}f' - \frac{1}{r^2}f = 0$

$$\Leftrightarrow f'' + \left(\frac{f}{r}\right)' = 0$$

$$f' + \frac{f}{r} = C_3 \iff rf' + f = C_3 r$$

$$\iff (rf)' = r C_3$$

$$rf = \frac{C_3}{2} r^2 + C_4$$

$$f(r) = \frac{C_3}{2} r + C_4 \frac{1}{r}$$

$$\varphi = \left(\frac{C_3}{2} r + C_4 \frac{1}{r} \right) \cos\theta$$

$r \to \infty$: $\varphi = \frac{C_3}{2} r \cos\theta = U r \cos\theta$

$$\Rightarrow C_3 = 2U$$

$r = a$: $\left(U - \frac{C_4}{a^2} \right) \cos\theta = 0$

$$\rightarrow C_4 = U a^2$$

$$\varphi = U \left(r + \frac{a^2}{r} \right) \cos\theta$$

$$\begin{cases} u_r = U \left(1 - \frac{a^2}{r^2} \right) \cos\theta \\ u_\theta = -U \left(1 + \frac{a^2}{r^2} \right) \sin\theta \end{cases}$$

When there is circulation round the cylinder, derive the equation

$$\frac{r}{a} = \frac{B}{2} + \left(\frac{B^2}{4} - 1\right)^{\frac{1}{2}}, \quad \theta = \frac{3\pi}{2},$$

and confirm that the stagnation points vary in position with the parameter B.

$$\bar{u} = \bar{u}_0 + \bar{u}_v$$

$$\bar{u}_v = \frac{\Gamma}{2\pi r}\bar{e}_\theta, \quad \Gamma < 0.$$

$$\begin{cases} u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \\ u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} \end{cases}$$

Stagn. points at $r = a$:

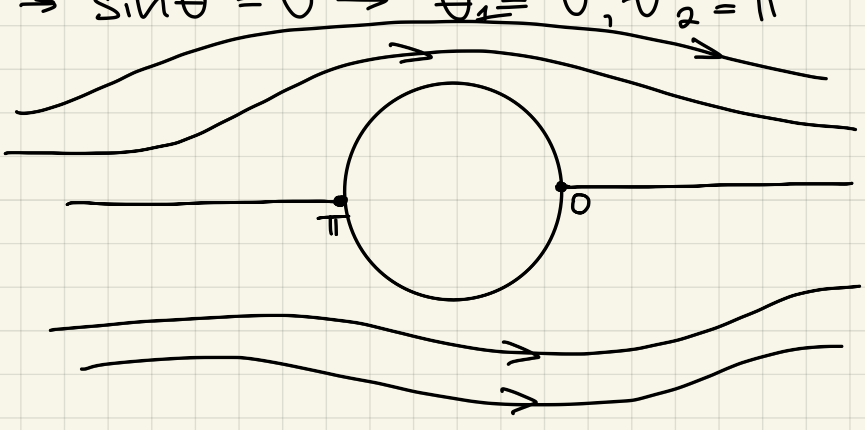
$$u_r = 0, \quad u_\theta = -2U\sin\theta + \frac{\Gamma}{2\pi a} = 0$$

$$\sin\theta = \frac{\Gamma}{4U\pi a} = -\frac{B}{2},$$

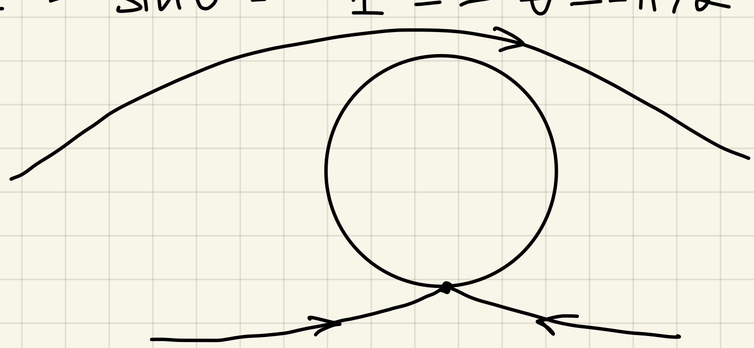
$$B = -\frac{\Gamma}{2U\pi a} > 0.$$

$$-1 \le \sin\theta \le 1 \Rightarrow \quad 0 < B \le 2$$

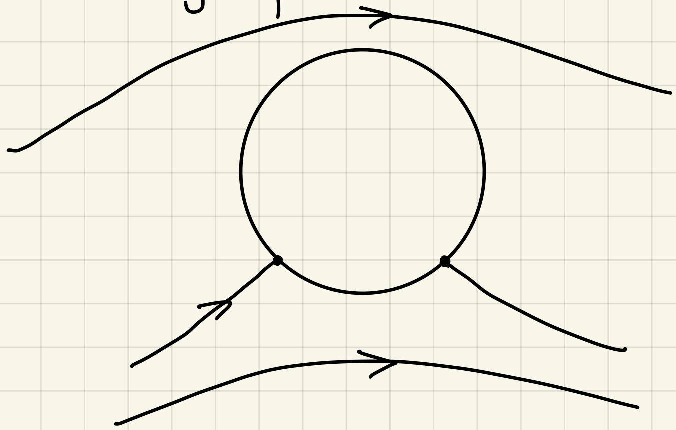
I. $B = 0 \to \sin\theta = 0 \to \theta_1 = 0, \theta_2 = \pi$



II. $B = 2 \to \sin\theta = -1 \Rightarrow \theta = -\pi/2$



III. $B < 2 \rightarrow$ 2 stagn. points.



What about stagnation points at $r > a$? This implies

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta = 0$$

$$\Rightarrow \cos\theta = 0 \rightarrow \theta = \pm \pi/2$$

If $\theta = \pi/2$,

$$u_\theta = -U\left(1 + \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi r} = 0$$

this is our eq-n for computing r.

Define $B = -\dfrac{\Gamma}{2\pi U a}$,

$$1 + \frac{a^2}{r^2} = -\frac{Ba}{r}$$

$$\rightarrow r^2 + Bar + a^2 = 0.$$

If $\theta = -\pi/2$, our quadratic eq-n is

$$r^2 - Bar + a^2 = 0$$

Solution: $\dfrac{r}{a} = \dfrac{B}{2} \pm \sqrt{\dfrac{B^2}{4} - 1}$.

We know that $r > a$

$$\rightarrow \frac{r}{a} = \frac{B}{2} + \sqrt{\frac{B^2}{4} - 1}$$

As r must be a real number,

$$\frac{B^2}{4} - 1 \geq 0$$

$$\rightarrow B^2 \geq 4.$$

By our choice, $B > 0 \Rightarrow B \geqslant 2$.

Moreover, as $r/a > 1$, $B > 2$.